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HOMOGENEOUS SOLUTIONS OF THE PROBLEM OF STEADY VIBRATIONS OF A PIEZOCERAMIC CYLINDER*

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Steady vibrations of a hollow piezoceramic cylinder with radial polarization are considered. An analytical and numerical analysis is performed of the homogeneous solutions, and the behaviour of the dispersion curves of the real and complex modes is investigated as a function of the geometrical parameters.

Wave propagation in a solid cylinder of electroelastic material with axial polarization was investigated earlier /1/ by the method of homogeneous solutions. Plane problems of the vibrations of piezoceramic cylinders with different types of polarization were examined in /2, 3/. Using a variational method, the vibrations of a finite cylinder were considered in /4/.

1. We consider steady axisymmetric vibrations of a piezoceramic cylinder polarized along the radius. The cylinder inner radius is r_1, the outer radius is r_2, and the length of the generatrix is $2l$.

We introduce the r, φ, z cylindrical coordinate system by directing the z axis along the cylinder axis.

We shall assume that the material properties are described by the following relationships /5/:

$$\sigma_{rr} = c_{33}^E\varepsilon_r + c_{13}^E(\varepsilon_\varphi + \varepsilon_z) - e_{33}E_r, \quad \sigma_{\varphi\varphi} = c_{13}^E\varepsilon_r + c_{11}^E\varepsilon_\varphi + c_{12}^E\varepsilon_z - e_{13}E_r \tag{1.1}$$

$$\sigma_{zz} = c_{13}^E\varepsilon_r + c_{12}^E\varepsilon_\varphi + c_{11}^E\varepsilon_z - e_{13}E_r, \quad \sigma_{rz} = 2c_{44}^E\varepsilon_{rz} - e_{15}E_z$$

$$D_r = e_{13}(\varepsilon_\varphi + \varepsilon_z) + e_{33}\varepsilon_r + \varepsilon_{33}^s E_r, \quad D_z = 2e_{15}\varepsilon_{rz} + \varepsilon_{11}^s E_z$$

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\varphi = \frac{u_r}{r}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

Here σ_{kl} are the stress tensor components, u_r, u_z, D_k, E_k are displacement vector components, electrical induction, and electric field strength, respectively, c_{ij}^E are the elastic moduli, e_{ij} the piezomoduli, and ε_{ij}^s the permittivities.

Adding the equations of motion

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} + \frac{\partial\sigma_{rz}}{\partial z} = \rho\frac{\partial^2 u_r}{\partial t^2}, \quad \frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = \rho\frac{\partial^2 u_z}{\partial t^2}$$

and the equations of forced dielectric electrostatics

$$\operatorname{div} D = 0, \quad \operatorname{rot} E = 0$$

to (1.1), we obtain the closed system of equations

$$(e^{-2\varepsilon\xi}L_0 + e^{-\varepsilon\xi}i\alpha L_1 - \alpha^2 L_2 + \Omega^2 L_3)v = 0 \tag{1.2}$$

Here

$$u = u(r)e^{i(kz-\omega t)}, \quad \psi = \psi(r)e^{i(kz-\omega t)}$$

$$\xi = \frac{1}{\varepsilon}\ln\frac{r}{r_0}, \quad \varepsilon = \frac{1}{2}\ln\frac{r_2}{r_1}, \quad r_0 = \sqrt{r_1r_2}, \quad \zeta = \frac{z}{r_0}, \quad \partial = \frac{\partial}{\partial\xi}, \quad \xi \in [-1, 1]$$

$$\Omega^2 = \frac{\rho r_0^2\varepsilon^2\omega^2}{c_0}, \quad \alpha = \varepsilon r_0 k, \quad v = \operatorname{col}\{u_r, u_z, \psi\}, \quad c_{ij}^* = \frac{c_{ij}^E}{c_0}, \quad e_{ij}^* = \frac{e_{ij}e_0}{c_0}, \quad \varepsilon_{ij}^* = \frac{\varepsilon_{ij}^s e_0^2}{c_0}$$

ψ is the electric field potential associated with the intensity vector by means of the relationship $E = -\operatorname{grad}\psi$, k is the wave number, ω is the vibration frequency, ρ is the density, and c_0, e_0 are certain characteristic parameters of the material which are of dimensionality $c_{ij}^s, |E|$. To simplify the writing, we omit the asterisk.

The matrix operators have the following form:

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$$L_0 = \left\| \begin{matrix} c_{33}\partial^2 - \varepsilon^2 c_{11} & 0 & e_{33}\partial^2 - \varepsilon e_{13}\partial \\ 0 & c_{41}\partial^2 & 0 \\ e_{33}\partial^2 + \varepsilon e_{13}\partial & 0 & -\varepsilon_{33}\partial^2 \end{matrix} \right\|, \quad L_2 = \left\| \begin{matrix} c_{44} & 0 & e_{15} \\ 0 & c_{11} & 0 \\ e_{15} & 0 & -\varepsilon_{11} \end{matrix} \right\|, \quad L_3 = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

$$L_1 = \left\| \begin{matrix} 0 & (c_{13} + c_{44})\partial + \varepsilon(c_{13} - c_{12}) & 0 \\ (c_{13} + c_{44})\partial + \varepsilon(c_{12} + c_{44}) & 0 & (e_{13} + e_{15})\partial + \varepsilon e_{15} \\ 0 & (e_{13} + e_{15})\partial + \varepsilon e_{13} & 0 \end{matrix} \right\|$$

We shall consider the side surfaces of the cylinder to be stress-free and without electrodes thereon

$$(e^{-\varepsilon\xi} M_0 + i\alpha M_1)\, v\, |_{\xi=\pm 1} = 0 \tag{1.3}$$

$$M_0 = \left\| \begin{matrix} c_{33}\partial + \varepsilon c_{13} & 0 & e_{33}\partial \\ 0 & c_{44}\partial & 0 \\ e_{33}\partial + \varepsilon e_{13} & 0 & -\varepsilon_{33}\partial \end{matrix} \right\|, \quad M_1 = \left\| \begin{matrix} 0 & c_{13} & 0 \\ c_{44} & 0 & e_{15} \\ 0 & e_{13} & 0 \end{matrix} \right\|$$

The dispersion curves of problem (1.2), (1.3) are constructed below by numerical methods.

2. We turn first to an analytical study of the spectral problem (1.2), (1.3) under the assumption that the parameter ε characterizing the cylinder thickness is sufficiently small.

We start the investigation with the static case $\Omega = 0$. Omitting intermediate calculations, we give a brief description of the spectrum.

In the nature of their behaviour in ε the set of eigenvalues can be separated into three groups. The first group consists of a double eigenvalue $\alpha^{(1)} = 0$. An eigenvector of the following from corresponds to it: $v^{(1)} = \mathrm{col}\{0, A_1, A_2\}$ (A_1, A_2 are arbitrary constants).

The second group consists of four zeros which tend to zero as $\varepsilon \to 0$. The first terms of their asymptotic expansion have the form

$$\alpha^{(2)} = \varepsilon^{1/2}[\alpha_0 + O(\varepsilon)]$$

$$\alpha_0^4 = 3(c_{12} - c_{11})\frac{2a_1 c_{13} - 2a_2 e_{13} - c_{11} - c_{12}}{(a_1 c_{13} + a_2 e_{13} - c_{11})^2} > 0, \qquad a_1 = \frac{c_{13}\varepsilon_{33} + e_{13}e_{33}}{c_{33}\varepsilon_{33} + e_{33}^2}, \quad a_2 = \frac{c_{13}e_{33} - c_{33}e_{13}}{c_{33}\varepsilon_{33} + e_{33}^2}$$

We write down the first term of the expansion in ε for the eigenvector $v^{(2)} = \mathrm{col}\{A, 0, 0\}$ (A is an arbitrary constant).

The third group of eigenvalues consists of a countable set of roots $\alpha_j^{(3)}$ ($j = 0, 1, \ldots$), with the following asymptotic representation: $\alpha_j^{(3)} = \beta_j + O(\varepsilon)$, where β_j are the eigenvalues of the corresponding spectral problem for a strip (layer) investigated in detail in /6, 7/.

The first group of eigenvalues determines the penetrating solution, whose state of stress and strain is equivalent to the principal vector of the forces P acting in the transverse section. It can be shown that if $P = 0$, the corresponding solution vanishes.

The state of stress and strain corresponding to the second group has the nature of an edge effect inherent in thin shells, and is equivalent to the action of a bending moment and a transverse force in the transverse section.

The state of stress and strain in the third group has the nature of a boundary layer localized at the cylinder endfaces.

3. We will now investigate the spectrum of problem (1.2), (1.3) for small α and Ω. An analysis performed on the structure of the static problem spectrum enables the form of the analytical expansions in Ω to be determined for different groups of zeros.

The eigenvalues corresponding to the first group can be sought in the form

$$\alpha = \Omega t_1 + \ldots, \quad v = v_0 + \Omega v_1 + \ldots \tag{3.1}$$

Substituting expansion (3.1) into (1.2) and (1.3) we obtain a certain recursion system, which when integrated yields

$$v_0 = \left\| \begin{matrix} 0 \\ B_1 \\ B_2 \end{matrix} \right\|, \quad v_1 = \left\| \begin{matrix} t_1 B_1 (b_1 e^{\varepsilon\nu\xi} + b_2 e^{-\varepsilon\nu\xi} + b_3 e^{\varepsilon\xi}) \\ -t_1 B_2 c_{44}^{-1} e_{15} e^{\varepsilon\xi} + D_1 \\ t_1 B_1 (b_4 e^{\varepsilon\nu\xi} + b_5 e^{-\varepsilon\nu\xi} + b_6 e^{\varepsilon\xi}) + D_2 \end{matrix} \right\| \tag{3.2}$$

$$\begin{aligned} t_1^2 = {}^1/_2\, \mathrm{sh}\, 2\varepsilon\, \{&[(1+\nu)\varepsilon_{33}]^{-1}[(c_{12} + \nu c_{13})\varepsilon_{33} + \\ &(e_{13} + \nu e_{33}) e_{13}]\, b_1 \mathrm{sh}\, \varepsilon(1+\nu) + [(1-\nu)\varepsilon_{33}]^{-1} \times \\ &[(c_{12} - \nu c_{13})\varepsilon_{33} + (e_{13} - \nu e_{33}) e_{13}]\, b_2\, \mathrm{sh}\, \varepsilon(1-\nu) + \\ &{}^1/_2 [(c_{12} + c_{13}) b_3 + e_{13} b_6 + c_{11}]\, \mathrm{sh}\, 2\varepsilon\}^{-1} \end{aligned}$$

$$\nu^2 = (c_{33}\varepsilon_{33} + e_{33}^2)^{-1}(c_{11}\varepsilon_{33} + e_{13}^2)$$

($b_1, b_2, \ldots, b_6$ depend on the constants of the material, which are not presented here because of the awkwardness of their expressions, and B_1, B_2 D_1, D_2 are arbitrary constants).

The relationships (3.1)', (3.2) describe the beginning of the first dispersion curve in the neighbourhood of the point $\alpha = 0, \Omega = 0$.

The beginning of the remaining dispersion curves is determined from the condition $\alpha = 0$. The determinant of the boundary conditions is here decomposed into two transcendental equations containing Bessel and Weber functions:

$$\mathrm{I}_1(\mathrm{q}e^{\varepsilon})\,\mathrm{Y}_1(\mathrm{q}e^{-\varepsilon}) - \mathrm{I}_1(\mathrm{q}e^{-\varepsilon})\,\mathrm{Y}_1(\mathrm{q}e^{\varepsilon}) = 0$$

$$\left[\mathrm{I}_\nu(\mathrm{s}e^{\varepsilon}) - \frac{\mathrm{s}e^{\varepsilon}}{\mathrm{k}\nu+1} I_{\nu+1}(\mathrm{s}e^{\varepsilon})\right]\left[Y_\nu(\mathrm{s}e^{-\varepsilon}) - \frac{\mathrm{s}e^{-\varepsilon}}{\mathrm{k}\nu+1} Y_{\nu+1}(\mathrm{s}e^{-\varepsilon})\right] -$$

$$\left[\mathrm{I}_\nu(\mathrm{s}e^{-\varepsilon}) - \frac{\mathrm{s}e^{-\varepsilon}}{\mathrm{k}\nu+1} \mathrm{I}_{\nu+1}(\mathrm{s}e^{-\varepsilon})\right]\left[\mathrm{Y}_\nu(\mathrm{s}e^{\varepsilon}) - \frac{\mathrm{s}e^{\varepsilon}}{\mathrm{k}\nu+1} \mathrm{Y}_{\nu+1}(\mathrm{s}e^{\varepsilon})\right] = 0$$

$$\mathrm{q} = \frac{\Omega}{\varepsilon\sqrt{c_{44}}},\quad \mathrm{s} = \frac{\Omega}{\varepsilon}\left(\frac{\varepsilon_{33}}{c_{33}\varepsilon_{33} + e_{33}^2}\right)^{1/2},\quad \mathrm{k} = a_1^{-1}$$

The two sets of roots obtained determine the initial points of the real curves on the Ω axis. The values of Ω corresponding to these points are the resonance frequencies of radial vibrations of an ininfite cylinder.

We examine the behaviour of the dispersion curves for α and Ω tending to infinity. We assume that the limit of their ratio remains a constant, i.e., $\lim \Omega/\alpha = \mathrm{const}$ as $\alpha \to \infty$, $\Omega \to \infty$. We convert problem (1.2), (1.3) to the form

$$(\mu^2 e^{-2\varepsilon\xi} L_0 + e^{-\varepsilon\xi} \mathrm{i}\mu L_1 - L_2 + C^2 L_3)\,\mathrm{v} = 0 \tag{3.3}$$

$$(\mu e^{-\varepsilon\xi} M_0 + \mathrm{i}M_1)\,\mathrm{v}\,|_{\xi=\pm1} = 0;\quad \mu = 1/\alpha,\quad C^2 = \Omega^2/\alpha^2$$

where C is a new spectral parameter corresponding to the wave propagation phase velocity. Evidently $\mu \to 0$ as $\alpha \to \infty$, i.e., μ is a small parameter. From the mechanics viewpoint, this corresponds to the case when the wavelength is considerably less than the thickness of the cylinder under consideration.

M.I. Vishik and L.A. Liusternik developed general methods for solving such problems, which consist of performing two iterations. To construct the internal solution, we execute the first iteration. Using the expansions

$$\mathrm{v} = \mathrm{v}_0 + \mu\mathrm{v}_1 + \ldots,\quad C^2 = C_0^2 + \mu C_1^2 + \ldots \tag{3.4}$$

we obtain a recurrent system. Its analysis shows that wave propagation is possible with two phase velocities

$$C_0^{(1)} = (c_{11}/\rho)^{1/2},\quad C_0^{(2)} = \left(\frac{c_{44} + e_{15}^2/\varepsilon_{11}}{\rho}\right)^{1/2}$$

We will now investigate problem (3.3) on the basis of the second iteration process. To do this, we stretch the scale in the neighbourhood of the boundary by introducing a new variable. The first stage of the iteration process yields

$$(L_0 + L_1 + L_2 + C_R^2 L_3)\,\mathrm{v} = 0,\quad (M_0 + M_1)\,\mathrm{v}\ (0) = 0$$

This boundary value problem together with the additional condition of the solution decreasing at infinity describes a wave propagating along a free surface; C_R is its phase velocity determined numerically.

4. We will present some results of a numerical analysis of problem (1.2), (1.3).

The investigation was performed for cylinders made of PZT-4 material which has first been polarized in the radial direction. The piezoceramic moduli are /8/

$$c_{11}^E = 13.9,\ c_{12}^E = 7.78,\ c_{13}^E = 7.43,\quad c_{33}^E = 11.5,\ c_{44}^E = 2.56,$$

$$e_{13} = -5.2,\ e_{33} = 15.1,\ e_{15} = 12.7,\ \varepsilon_{11}^S/\varepsilon_0^S = 730,\ \varepsilon_{33}^S/\varepsilon_0^S = 635,$$

$$[c_{ij}^S] = 10^{10}\,\mathrm{N/m^2},\ [e_{ij}] = \mathrm{K/m^2},\ \varepsilon_0^S = 8.85\cdot10^{-12}\ \Phi/\mathrm{m},\ \rho = 7.5\cdot10^3\ \mathrm{kg/m^3}$$

Cylinders with $\varepsilon = 1.151$; $\varepsilon = 0.053$; $\varepsilon = 0.001$ were examined.

The set of dispersion curves $\alpha(\Omega)$ was divided into two parts, real and complex, in a numerical analysis whose basis was Godunov's method /9/ of orthogonal factorization. The pure imaginary eigenvalues α were also referred to the complex. Godunov's method in conjunction with the argument method was used to find the complex zeros.

Dispersion curves for the cylinders with $\varepsilon = 1.151$ (Fig.1) and $\varepsilon = 0.053$ (Fig.2) are represented in the graphs. The dispersion curves for $\varepsilon = 0.001$ and $\varepsilon = 0.053$ are practically coincident, with the exception of a small neighbourhood of the origin. Therefore, for $\varepsilon < 0.06$ the curvature of the cylindrical surfaces exert no substantial influence on the strucutre of the curves.

The real and pure imaginary values of α are superposed by solid lines on the graphs, where the pure imaginary roots are laid off to the left of the origin. The real and imaginary parts of the complex branches are superposed by dashes; here OX is the axis of real values of α and OY is the axis of imaginary values of α. Values of the dimensionsless frequency Ω are plotted along the OZ axis.

For a fixed value of Ω the complete solution for a finite cylinder can be represented in the form

$$\mathbf{v} = \sum_{j=1}^{n} A_j \mathbf{v}_j(\xi)\, e^{i(a_j\zeta - \Omega\tau)} + \sum_{k=1}^{\infty} B_k v_k(\xi)\, e^{i(\alpha_k\zeta - \Omega\tau)} \tag{4.1}$$

where α_k are complex eigenvalues, $a_k = \mathrm{Re}\,\alpha_k > 0$, $\mathbf{v}_k$ are the corresponding eigenvectors, and A_j, B_j are arbitrary constants determined when the boundary conditions on the endfaces are satisfied.

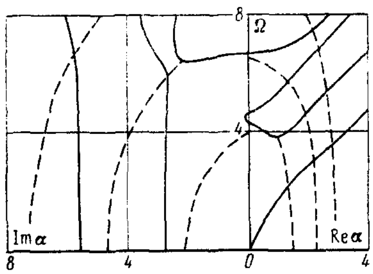

Fig.1

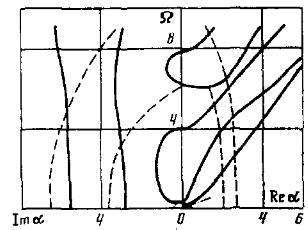

Fig.2

The first sum determines the penetrating solution. Real dispersion curves on the graphs correspond to it. The second sum determines the boundary layer solution localized at the cylinder endfaces. The dispersion curves from the left half-planes of the graphs correspond to it. It is seen that the complex roots can have a small imaginary part for certain Ω. In this case the corresponding homogeneous boundary layer solutions can exert a singificant influence on the state of stress and strain in the inner part of the cylinder.

In the case of an infinite cylinder, there is no second sum in (4.1), and the first describes a wave propagating in the cylinder. The number of such waves is determined by the quantity Ω and the presence of vibrations sources at infinity. If the sources are arranged for $\zeta = -\infty$, then only $a_k > 0$ are taken, if there are still sources at $\zeta = +\infty$, then the summation is over all a_k, both positive and negative.

The dash-dot lines in the graphs denote the asymptotic values from (3.1), which approximate the first dispersion curve well for low frequencies.

In the high-frequency case and short wavelengths compared with the cylinder thickness, the dispersion curves emerge on the asyptote (3.4). The phase velocity for the first curve tends to the surface-wave phase velocity, and to the shear wave velocity for the remaining curves.

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